

Combinatorial Excess Intersection

Jose Rodriguez*

December 12, 2012

Abstract

We provide formulas and algorithms for computing the excess numbers of certain ideals. The solution for monomial ideals is given by the mixed volumes of certain polytopes. These results enable us to design specific homotopies for numerical algebraic geometry.

1 Introduction

Consider a homogeneous ideal $\mathcal{I} \subset \mathbb{C}[x_0, \dots, x_n]$ generated by the forms B_1, \dots, B_k of degrees p_1, \dots, p_k . Let f_1, \dots, f_n be forms of \mathcal{I} of degrees d_1, \dots, d_n such that $d_i > p_j$ for all i, j . Since each $f_i \in \mathcal{I}$ we have $(f_1, \dots, f_n) \subset \mathcal{I}$ and $\mathbf{V}(f_1, \dots, f_n) \supset \mathbf{V}(\mathcal{I})$. The *excess intersection* of (f_1, f_2, \dots, f_n) with respect to \mathcal{I} is defined as the quasiprojective variety $\mathbf{V}(f_1, \dots, f_n) \setminus \mathbf{V}(\mathcal{I})$. We define the *excess number* $E_\bullet(\mathcal{I}; d_1, \dots, d_n)$ of an ideal \mathcal{I} to be the number of isolated solutions in $\mathbf{V}(f_1, \dots, f_n) \setminus \mathbf{V}(\mathcal{I})$.

In this paper, we consider the case when the polynomials f_1, \dots, f_n are chosen to be \mathcal{I} -generic forms (Definition 2), which has applications in machine learning and ideal regression. When the generators of the ideal $\mathcal{I} \subset \mathbb{C}[x_0, \dots, x_n]$ are all generic forms or all monomials, the excess intersection of (f_1, \dots, f_n) is a finite number of reduced points in \mathbb{P}^n whose cardinality is determined by d_1, \dots, d_n [1, 2]. The cardinality of this set of points is exactly the excess number

$$E_\bullet(\mathcal{I}; d_1, \dots, d_n) = \#\mathcal{V}(f_1, \dots, f_n) \setminus \mathcal{V}(\mathcal{I}).$$

At times it will be more convenient to work with the number $E_o(\mathcal{I}; d_1, \dots, d_n) := d_1 \cdots d_n - E_\bullet(\mathcal{I}; d_1, \dots, d_n)$ for this case. This number is classically known as the *equivalence* of an ideal [3] [Chapter 6]. We will see how $E_\bullet(\mathcal{I}; d_1, \dots, d_n)$ and $E_o(\mathcal{I}; d_1, \dots, d_n)$ relate to the volume of a subdivided simplex. Our main contribution is a combinatorial proof of the theorem below, and algorithms using numerical algebraic geometry to compute excess numbers of any ideal.

Theorem 1. *Let \mathcal{I} be an ideal of $\mathbb{C}[x_0, \dots, x_n]$ generated by generic forms B_1, \dots, B_k of degree p_1, \dots, p_k . If f_1, \dots, f_n are \mathcal{I} -generic forms of degree d_1, \dots, d_n , then*

$$E_\bullet(\mathcal{I}; d_1, \dots, d_n) + p_1 \cdots p_k \sum_{\delta=0}^{n-k} \left((-1)^\delta \mathcal{D}_{n-k-\delta} \mathcal{P}_\delta \right) = d_1 d_2 \cdots d_n$$

where $\mathcal{D}_{n-k-\delta}$ is the sum of all square-free monomials in d_1, \dots, d_n of degree $n - k - \delta$ and \mathcal{P}_δ is the sum of all monomials in p_1, \dots, p_k of degree δ .

*The author is supported by the US National Science Foundation DMS-0943745.

The paper is structured as follows. We consider the case when \mathcal{I} is a monomial ideal, and show excess numbers equal mixed volumes of certain polytopes (Lemma 6). By further restricting to the case when the ideal \mathcal{I} defines a complete intersection that is also a linear space (though not necessarily reduced), we do a mixed volume computation (Lemma 9) to get an explicit formula for excess numbers. By Proposition 5 and Lemma 10, we immediately get the main result in Theorem 1.

While also having applications to machine learning [1, 2], the motivation for this work came from Mike Stillman at the 2012 Institute for Mathematics and its Applications Participating Institution Summer Program for Graduate Students in Algebraic Geometry for Applications when he proposed the following problem. Given an ideal defining a curve, can we determine its equivalence? We give an answer to this question when the ideal is generated by generic forms and when the ideal is generated by monomials. Our computations were performed with the software `bertini` and `Macaulay2` [4, 5]. In addition, we remark that our work has direct relations to computing Segre classes as seen in [6, 7, 8].

2 The Monomial Case

The key idea to Theorem 1 is to cast our excess intersection problem in the language of combinatorial geometry and prove Proposition 5. Since the following proofs will use Newton polytopes, Minkowski sums, and genericity, we set up additional notation here.

The *Newton polytope* of a form f will be denoted as $\mathcal{N}(f)$. The standard n -simplex is the convex hull of the origin ϵ_0 , and the standard basis of unit vectors $\epsilon_1, \dots, \epsilon_n$ in \mathbb{R}^n . The binary operation, Minkowski sum, will be denoted as “+”. We take a *generic form* to be a homogeneous polynomial with full support and random coefficients. For example $c_1x + c_2y + c_3z + c_4w$ is a generic linear form in $\mathbb{C}[x, y, z, w]$ whenever the c_i are chosen to be random numbers. For a more detailed definition of generic form, we refer to [1]. We now give a precise definition of a system of \mathcal{I} -generic forms.

Definition 2. Let \mathcal{I} be an ideal of $\mathbb{C}[x_0, \dots, x_n]$ generated by forms B_1, \dots, B_l whose respective degrees are p_1, \dots, p_l . If the forms f_1, \dots, f_n are defined by

$$\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1l} \\ a_{21} & \cdots & a_{2l} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nl} \end{bmatrix} \begin{bmatrix} B_1 \\ \vdots \\ B_l \end{bmatrix},$$

then f_1, \dots, f_n are said to be a system of *\mathcal{I} -generic forms* with degrees d_1, \dots, d_n whenever a_{ij} are generic forms with positive degree equal to $d_i - \deg B_j$.

With this definition, the degree of each f_i is always greater than the degree of B_j for all i, j . It is important to note the difference between a *form with generic coefficients*, an *\mathcal{I} -generic form*, and a *generic form*. A *form with generic coefficients* is a homogeneous polynomial with random coefficients that does not necessarily have full support. For example, whenever c_1, c_2 are random coefficients, the form $c_1x + c_2y \in \mathbb{C}[x, y, z, w]$ has generic coefficients, but is not a generic form. An important fact is that an \mathcal{I} -generic form need not have generic coefficients! For example, consider the ideal $\mathcal{I} = (x + y)$ of $\mathbb{C}[x, y, z, w]$. Then, an \mathcal{I} -generic form of degree 2 may be

written as

$$(c_1x + c_2y + c_3z + c_4w)(x + y) = c_1x^2 + c_2y^2 + (c_1 + c_2)xy + c_3(zx + zy) + c_4(wx + wy).$$

Since the coefficients of zw and zy are equal (as well as the coefficients of wx and wy), this \mathcal{I} -generic form does not have generic coefficients. But, whenever \mathcal{I} is a monomial ideal, an \mathcal{I} -generic form will always have generic coefficients. This is key to proving Lemma 6.

Remark 3. As mentioned in the introduction, a system f_1, \dots, f_n of n \mathcal{I} -generic forms of $\mathbb{C}[x_0, \dots, x_n]$ has a variety consisting of $\mathbf{V}(\mathcal{I})$ and a union of finitely many points whenever the degrees of each f_i are greater than the degrees of the generators of \mathcal{I} . Also, if we take the \mathcal{I} -generic forms f_1, \dots, f_{n+l} with $\mathcal{I} \subset \mathbb{C}[x_0, \dots, x_n]$ and $l > 0$, then $\mathbf{V}(f_1, \dots, f_{n+l})$ is equal to $\mathbf{V}(\mathcal{I})$ [1, 2].

Whenever \mathcal{I} is a principal ideal, the excess numbers are easy to determine.

Example 4. Let $\mathcal{I} = (B)$ be a principal ideal of the ring $\mathbb{C}[x_0, \dots, x_n]$ with $\deg B = p$. Then n \mathcal{I} -generic forms may be written as

$$f_1 = a_1B, f_2 = a_2B, \dots, f_n = a_nB,$$

with $\deg f_i = d_i$ and a_i being a generic form of degree $d_i - p$. To determine the excess number $E_\bullet(\mathcal{I}; d_1, \dots, d_n)$, we saturate (f_1, \dots, f_n) by \mathcal{I} . Doing so, we conclude that the excess intersection of (f_1, \dots, f_n) is defined by (a_1, \dots, a_n) and consists of finitely many points. By Bezout's theorem, it follows that $E_\bullet(\mathcal{I}; d_1, \dots, d_n) = (d_1 - p)(d_2 - p) \cdots (d_n - p)$.

In general, determining the defining equations of an excess intersection is harder. But Lemma 6 allows us to determine excess numbers of monomial ideals directly using combinatorics. As a consequence, we can prove the following proposition, getting an explicit formula for excess numbers of certain ideals.

Proposition 5. Let \mathcal{I} be an ideal of $\mathbb{C}[x_0, \dots, x_n]$ generated by generic forms $x_1^{p_1}, \dots, x_k^{p_k}$. If f_1, \dots, f_n are \mathcal{I} -generic forms of degree d_1, \dots, d_n , then

$$E_\bullet(\mathcal{I}; d_1, \dots, d_n) + p_1 \cdots p_k \sum_{\delta=0}^{n-k} \left((-1)^\delta \mathcal{D}_{n-k-\delta} \mathcal{P}_\delta \right) = d_1 d_2 \cdots d_n$$

where $\mathcal{D}_{n-k-\delta}$ is the sum of all square-free monomials in d_1, \dots, d_n of degree $n - k - \delta$ and \mathcal{P}_δ is the sum of all monomials in p_1, \dots, p_k of degree δ .

To prove Proposition 5, we first prove Lemma 6. This lemma shows that excess numbers of any monomial ideal can be computed using Bernstein's theorem [9] and mixed volumes. After this, we set up and do a mixed volume computation in Lemma 8 and Lemma 9. The results of this mixed volume immediately give us Proposition 5.

Lemma 6. Let $\mathcal{I} \subset \mathbb{C}[x_0, \dots, x_n]$ be a monomial ideal, and let f_1, \dots, f_n be \mathcal{I} -generic forms of degrees d_1, \dots, d_n . Then the excess number $E_\bullet(\mathcal{I}; d_1, \dots, d_n)$ equals the mixed volume of the Newton polytopes $\mathcal{N}(f_1), \dots, \mathcal{N}(f_n)$.

Proof. By Bernstein's theorem [9] the number of isolated solutions of f_1, \dots, f_n in \mathbb{P}^n with non-zero coordinates equals the mixed volume of the Newton polytopes of f_1, \dots, f_n whenever the forms f_1, \dots, f_n have generic coefficients. Because \mathcal{I} is a monomial ideal, the forms f_1, \dots, f_n indeed have generic coefficients. So we need only show that an isolated solution of the forms f_1, \dots, f_n has nonzero coordinates.

By way of contradiction, suppose $y = (y_0, \dots, y_n)$ is an isolated solution of the system f_1, \dots, f_n with zero for at least one coordinate. Without loss of generality we may assume $y_n = 0$. Let $\bar{\cdot}$ denote passing to the quotient ring $\mathbb{C}[x_0, \dots, x_n]/(x_n) \cong \mathbb{C}[x_0, \dots, x_{n-1}]$. Then $(y_0, y_1, \dots, y_{n-1}) \in \mathbb{P}^{n-1}$ is a solution to the system $\bar{f}_1, \dots, \bar{f}_n$ of $\bar{\mathcal{I}}$ -generic forms of $\mathbb{C}[x_0, \dots, x_{n-1}]$. But by Remark 3, there cannot be any isolated solutions to a system of n $\bar{\mathcal{I}}$ -generic forms in the ring $\mathbb{C}[x_0, \dots, x_{n-1}]$, a contradiction. \square

The key idea of this proof was to note the linear space defined by the generators of \mathcal{I} are contained in the coordinate planes. Thus, these hyperplanes are “hidden” and Bernstein's theorem counts exactly the solutions we are interested in. So to determine excess numbers of monomial ideals, we determine mixed volumes. In general, mixed volume computations are complicated, but in certain cases there is hope for an explicit formula. The certain case we now consider is when $\mathcal{I} = (x_1^{p_1}, \dots, x_k^{p_k})$ is generated by powers of unknowns. If f_1, \dots, f_n are \mathcal{I} -generic forms with degrees d_1, \dots, d_n , then we determine the excess number $E_\bullet(\mathcal{I}; d_1, \dots, d_n)$ by calculating the mixed volume of $\mathcal{N}(f_1), \dots, \mathcal{N}(f_n)$. Recall that the mixed volume [10] [Chapter 8.5] can be calculated by determining the coefficient of $\lambda_1 \lambda_2 \cdots \lambda_n$ in the polynomial defining the volume of the scaled Minkowski sum

$$\lambda_1 \mathcal{N}(f_1) + \cdots + \lambda_n \mathcal{N}(f_n).$$

If we let \mathfrak{S} denote a certain simplex, then Lemma 8 shows that slicing \mathfrak{S} by a certain hyperplane, we can subdivide \mathfrak{S} into two convex polytopes \mathfrak{S}_0 and \mathfrak{S}_1 . We will choose the certain hyperplane so that $\mathfrak{S}_1 = \lambda_1 \mathcal{N}(f_1) + \cdots + \lambda_n \mathcal{N}(f_n)$. Since $\text{Vol } \mathfrak{S} = \text{Vol } \mathfrak{S}_0 + \text{Vol } \mathfrak{S}_1$, we compute our desired mixed volume by determining the coefficients of $\lambda_1 \cdots \lambda_n$ in $\text{Vol } \mathfrak{S}_0$ and $\text{Vol } \mathfrak{S}$ (Lemma 9).

To elucidate our ideas we consider the following example.

Example 7. Let $\mathcal{I} = (x_1^{p_1}, x_2^{p_2})$ be an ideal of the ring $\mathbb{C}[x_0, x_1, x_2, x_3]$ and let f_1, f_2, f_3 be \mathcal{I} -generic forms of degrees d_1, d_2, d_3 . Then the Newton polytope of f_i will be the convex hull of two tetrahedra. Specifically, the Newton polytope of f_i is the convex hull of the following eight points in \mathbb{R}^3 of which six are vertices of $\mathcal{N}(f_i)$:

$$\begin{array}{cccc} (p_1, 0, 0) & (d_i, 0, 0) & (p_1, d_i - p_1, 0) & (p_1, 0, d_i - p_1) \\ (0, p_2, 0) & (0, d_i, 0) & (p_2, d_i - p_2, 0) & (p_2, 0, d_i - p_2). \end{array}$$

To avoid confusion with points in \mathbb{P}^n , we describe points in \mathbb{R}^3 as (u_1, u_2, u_3) rather than (x_1, x_2, x_3) . We may also describe the Newton polytope $\mathcal{N}(f_i)$ by its 5 supporting hyperplanes:

- the 3 coordinate hyperplanes,
- the hyperplane defined by $u_1 + u_2 + u_3 - d_i$, and
- the hyperplane defined by $\frac{u_1}{p_1} + \frac{u_2}{p_2} - 1$.

The normal vectors of the 5 hyperplanes supporting $\mathcal{N}(f_i)$ are the same for every i . Indeed they are the standard unit vectors $\epsilon_1, \epsilon_2, \epsilon_3$, the vector $\epsilon_1 + \epsilon_2 + \epsilon_3$, and the vector $\frac{1}{p_1}\epsilon_1 + \frac{1}{p_2}\epsilon_2$. By standard polytope theory [11][Proposition 7.12], it follows that the scaled Minkowski sum

$\lambda_1 \mathcal{N}(f_1) + \lambda_2 \mathcal{N}(f_2) + \lambda_3 \mathcal{N}(f_3)$, has the same 5 normal vectors as those of $\mathcal{N}(f_i)$. Indeed, the supporting hyperplanes are

- the 3 coordinate hyperplanes,
- the hyperplane defined by $u_1 + u_2 + u_3 - (\lambda_1 d_1 + \lambda_2 d_2 + \lambda_3 d_3)$, and
- the hyperplane defined by $\frac{u_1}{p_1} + \frac{u_2}{p_2} - (\lambda_1 + \lambda_2)$.

Now note that four of the five hyperplanes are defining facets of a simplex whose volume is $\frac{1}{3!} (\lambda_1 d_1 + \lambda_2 d_2 + \lambda_3 d_3)^3$. The fifth hyperplane subdivides the simplex as seen in Figure 1. By subtracting the volume of the white figure from the volume of the simplex, we attain the mixed volume by considering the coefficients of $\lambda_1 \lambda_2 \lambda_3$ in the difference. In this example, we would find the excess number satisfies $E_\bullet(x_1^{p_1}, x_2^{p_2}; d_1, d_2, d_3) + p_1 p_2 (d_1 + d_2 + d_3 - p_1 - p_2) = d_1 d_2 d_3$.

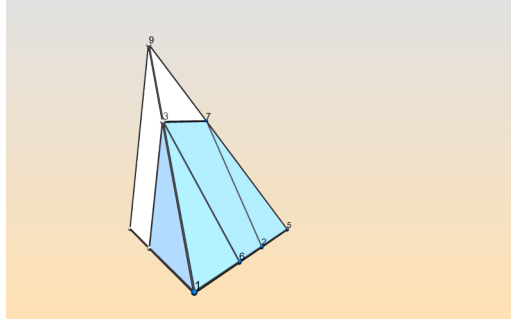


Figure 1:

We now precisely define the polytopes $\mathfrak{S}, \mathfrak{S}_0, \mathfrak{S}_1$. Let $D = \lambda_1 d_1 + \dots + \lambda_n d_n$ and $\Lambda = \lambda_1 + \dots + \lambda_n$. Then \mathfrak{S} is defined as the n -simplex whose $n+1$ vertices are the origin and $D\epsilon_i$. Moreover, the volume of \mathfrak{S} equals $D^n/n!$ which has a term $d_1 d_2 \dots d_n \lambda_1 \lambda_2 \dots \lambda_n$ when expanded out. If we define the hyperplane h_p by $\frac{u_1}{p_1} + \dots + \frac{u_k}{p_k} - \Lambda$, then h_p slices \mathfrak{S} into two convex polytopes \mathfrak{S}_0 containing the origin, and \mathfrak{S}_1 . We will prove that \mathfrak{S}_1 is the scaled Minkowski sum $\lambda_1 \mathcal{N}(f_1) + \dots + \lambda_n \mathcal{N}(f_n)$.

Lemma 8. *Let f_1, \dots, f_n be \mathcal{I} -generic forms with degrees d_1, \dots, d_n . If $\mathcal{I} = (x_1^{p_1}, \dots, x_k^{p_k})$, then $\mathfrak{S}_1 = \lambda_1 \mathcal{N}(f_1) + \dots + \lambda_n \mathcal{N}(f_n)$.*

Proof. We will show that the polytopes \mathfrak{S}_1 and $\lambda_1 \mathcal{N}(f_1) + \dots + \lambda_n \mathcal{N}(f_n)$ have the same $n+2$ supporting hyperplanes so that they must be equal. Specifically, we show that the supporting hyperplanes are

- the n coordinate hyperplanes,
- the hyperplane h_d defined by $u_1 + \dots + u_n - D$, and
- the hyperplane h_p defined by $\frac{u_1}{p_1} + \dots + \frac{u_k}{p_k} - \Lambda$.

The first $n+1$ hyperplanes support the simplex \mathfrak{S} . The last hyperplane h_p slices \mathfrak{S} into two polytopes one containing the origin and a second which is by definition the polytope \mathfrak{S}_1 . Now because $\lambda_i \mathcal{N}(f_i)$ has $n+2$ supporting hyperplanes consisting of

- the n coordinate hyperplanes,
- the hyperplane defined by $u_1 + \dots + u_n - \lambda_i d_i$, and
- the hyperplane defined by $\frac{u_1}{p_1} + \dots + \frac{u_k}{p_k} - \lambda_i$.

By standard polytope theory [11][Proposition 7.12], it follows $\lambda_1 \mathcal{N}(f_1) + \dots + \lambda_n \mathcal{N}(f_n)$ has the desired supporting hyperplanes. \square

With Lemma 8, we are able to calculate the mixed volume by determining the coefficient of $\lambda_1 \lambda_2 \cdots \lambda_n$ in a certain integral as seen in Lemma 9.

Lemma 9. *With the previous notation, $\text{Vol}(\mathfrak{S}_0)$ is a polynomial whose coefficient of $\lambda_1 \lambda_2 \cdots \lambda_n$ equals $p_1 \cdots p_k \sum_{\delta=0}^{n-k} \left((-1)^\delta \mathcal{D}_{n-k-\delta} \cdot \mathcal{P}_\delta \right)$.*

Proof. By Lemma 8, it follows that the volume of \mathfrak{S}_0 equals

$$\text{Vol}(\mathfrak{S}_0) = \int \left[\int \cdots \int d_{x_n} d_{x_{n-1}} \cdots d_{x_{k+1}} \right] d_\Delta$$

with the bounds of each integral inside the brackets with respect to d_{x_i} being $(x_i = 0) \rightarrow (x_i = D - x_{i-1} - \cdots - x_1)$ and Δ denotes the simplex in k -dimensional space with $k+1$ vertices of $\epsilon_0, \Lambda p_1 \cdot \epsilon_1, \dots, \Lambda p_k \cdot \epsilon_k$.

By using the calculus fact $\left[\int \cdots \int d_{x_n} d_{x_{n-1}} \cdots d_{x_{k+1}} \right] = \frac{1}{r!} (D - x_k - \cdots - x_1)^r$ and the binomial theorem, we have

$$\begin{aligned} \text{Vol}(\mathfrak{S}_0) &= \frac{1}{r!} \int (D - x_k - \cdots - x_1)^r d_\Delta \\ &= \frac{1}{r!} \sum_{\delta=0}^r \binom{r}{\delta} D^{r-\delta} \int (x_k + \cdots + x_1)^\delta d_\Delta \\ &= p_1 \cdots p_k \sum_{\delta=0}^r \left((-1)^\delta \frac{D^{n-k-\delta}}{(n-k-\delta)!} \frac{\Lambda^{k+\delta}}{(\delta+k)!} \mathcal{P}_\delta \right) \end{aligned}$$

with $r = n - k$. It is known how to integrate a linear form raised to some power over the simplex. So to get the last equality, we use [12] [Remark 9], that says $\int (x_k + \cdots + x_1)^\delta d_\Delta = \Lambda^{k+\delta} p_1 \cdots p_k \frac{\delta!}{(\delta+k)!} \mathcal{P}_\delta$.

Now, note that $L_m(\lambda_1, \dots, \lambda_n) := m!$ (the monomials in $\lambda_1, \dots, \lambda_n$ of degree m) is congruent to Λ^m modulo $\lambda_1^2, \dots, \lambda_n^2$. Similarly, note that D^m is congruent to $L_m(\lambda_1 d_1, \dots, \lambda_n d_n)$ modulo $\lambda_1^2, \dots, \lambda_n^2$. So we have

$$\begin{aligned} \text{Vol}(\mathfrak{S}_0) &\equiv p_1 \cdots p_k \sum_{\delta=0}^{n-k} \left((-1)^\delta L_{n-k-\delta}(\lambda_1 d_1, \dots, \lambda_n d_n) \cdot L_{k+\delta}(\lambda_1, \dots, \lambda_n) \mathcal{P}_\delta \right) \\ &\equiv \lambda_1 \cdots \lambda_n \cdot p_1 \cdots p_k \sum_{\delta=0}^{n-k} \left((-1)^\delta \mathcal{D}_{n-k-\delta} \mathcal{P}_\delta \right). \end{aligned}$$

The last congruence is shown by an easy combinatorial argument. \square

With Lemma 9, we have provided the explicit formula in Proposition 5. Since $\mathcal{I}' = (x_1^{p_1}, \dots, x_k^{p_k})$ is a complete intersection, by perturbing the defining equations of \mathcal{I}' by an ϵ -amount, we create an ideal \mathcal{I} which is a complete intersection defined by generic polynomials of $\mathbb{C}[x_0, \dots, x_n]$. Since we have perturbed the defining equations of \mathcal{I} by an ϵ -amount the isolated solutions in the excess of (f_1, \dots, f_n) with respect to \mathcal{I} will also be perturbed by an order of ϵ -amount. The point being that the number of isolated solutions is preserved and these two ideals have the same excess number giving us the following.

Lemma 10. *Let B_1, \dots, B_k be generic forms of $\mathbb{C}[x_0, \dots, x_n]$ of degree p_1, \dots, p_k . If $\mathcal{I}_1 = (x^{p_1}, \dots, x^{p_k})$, then $E_\bullet(\mathcal{I}_1; d_1, \dots, d_n) = E_\bullet(\mathcal{I}; d_1, \dots, d_n)$.*

As an immediate result of Lemma 10 and Proposition 5, we have Theorem 1.

Remark 11. We remark that the number $E_{\bullet}(\mathcal{I}; d_1, \dots, d_n)$ depends only on the Newton polytopes $\mathcal{N}(f_1), \dots, \mathcal{N}(f_n)$. For example, consider the ideals $\mathcal{I}_1 = (x^3, y^3)$, $\mathcal{I}_2 = (x^3, y^3, x^2y, xy^2)$, $\mathcal{I}_3 = (x^3, y^3, x^2y^2)$ in the ring $\mathbb{C}[w, x, y, z]$. All three of these ideals have the same excess intersection when every d_i is greater than 4 because the Newton polytopes of \mathcal{I}_i -generic forms are the same for $i = 1, 2, 3$. In particular, $E_{\bullet}(\mathcal{I}_i; 5, 5, 5) = 44$ for $i = 1, 2, 3$. But if we consider the ideal $\mathcal{J} = (x^3, y^3) + (xy)$, we find the Newton polytopes of \mathcal{J} -generic forms are different from those Newton polytopes of \mathcal{I}_i -generic forms. In particular, one can compute the excess number $E_{\bullet}(\mathcal{J}; 5, 5, 5)$ to be 65.

3 Algorithms from Numerical Algebraic Geometry

We have given a combinatorial description of excess numbers of monomial ideals in the first part of the paper and used this idea to give an explicit formula in Theorem 1. In the last part of this paper, we give algorithms that use homotopy continuation, an idea from numerical algebraic geometry to compute excess numbers of any ideal $\mathcal{I} \subset \mathbb{C}[x_0, \dots, x_n]$.

We set the notation for this section. Let $\mathcal{I} \subset \mathbb{C}[x_0, \dots, x_n]$ be an ideal generated by the forms B_1, \dots, B_l , and let $\mathcal{I}' \subset \mathbb{C}[x_0, \dots, x_n]$ be a monomial ideal generated by terms A_j where j is over some index set. Let f_1, \dots, f_n be \mathcal{I} -generic forms, and let f'_1, \dots, f'_n be \mathcal{I}' -generic forms such that $d_i = \deg f_i = \deg f'_i$. We implicitly assume that d_i is greater than $\deg B_j$ and $\deg A_{j'}$ for all possible i, j, j' .

Our algorithms will construct two homotopies, called \mathbf{h}_{upp} and \mathbf{h}_{ite} , that take the isolated solutions of the \mathcal{I}' -generic forms f'_1, \dots, f'_n as start points and tracks them to solutions of f_1, \dots, f_n giving bounds on $E_{\bullet}(\mathcal{I}; d_1, \dots, d_n)$. In the first algorithm, the monomial ideal \mathcal{I}' is constructed so that $E_{\bullet}(\mathcal{I}'; d_1, \dots, d_n) \geq E_{\bullet}(\mathcal{I}; d_1, \dots, d_n)$. By doing a numerical membership test [10] [Chapter 15], we will determine $E_{\bullet}(\mathcal{I}; d_1, \dots, d_n)$ and isolated solutions of $\mathbf{V}(f_1, \dots, f_n) \setminus \mathbf{V}(\mathcal{I})$ explicitly. In the second algorithm, the monomial ideal \mathcal{I}' is constructed to give lower bounds of $E_{\bullet}(\mathcal{I}; d_1, \dots, d_n)$ instead. But by iterating the second algorithm, we have a probabilistic way to make this bound sharp and compute all isolated solutions of $\mathbf{V}(f_1, \dots, f_n) \setminus \mathbf{V}(\mathcal{I})$ explicitly. The \mathbf{h}_{upp} -homotopy gets its name because it produces an *upper* bound of $E_{\bullet}(\mathcal{I}; d_1, \dots, d_n)$ prior to a membership test. The \mathbf{h}_{ite} -homotopy gets its name because several *iterations* can produce sharp lower bounds of $E_{\bullet}(\mathcal{I}; d_1, \dots, d_n)$ after a membership test.

3.1 Algorithm one and the \mathbf{h}_{upp} -homotopy

We now give a definition of the \mathbf{h}_{upp} -homotopy and prove that it does indeed provide an upper bound of $E_{\bullet}(\mathcal{I}; d_1, \dots, d_n)$ prior to a membership test.

Definition 12. Let $B_1, \dots, B_l \in \mathbb{C}[x_0, \dots, x_n]$ be forms such that $B_j = \sum_k A_{j,k}$ with $A_{j,k}$ being a term. (Note that $A_{j,k}$ has both a coefficient and a monomial associated to it.) To ease notation, let $\overrightarrow{A_j} = [A_{j,1}, \dots, A_{j,k_j}]$ be a row vector, $\overrightarrow{\alpha_{i,j}}$ be a row vector of k_j different generic forms, and $\overrightarrow{\beta_{i,j}}$ be a row vector of a generic form repeated k_j times. Define the \mathbf{h}_{upp} -homotopy

as $\mathbf{h}_{\text{upp}}(t; d_1, \dots, d_n) :=$

$$\left(t \begin{bmatrix} \overrightarrow{\alpha_{1,1}} & \overrightarrow{\alpha_{1,2}} & \cdots & \overrightarrow{\alpha_{1,l}} \\ \vdots & & & \vdots \\ \overrightarrow{\alpha_{n,1}} & \overrightarrow{\alpha_{n,2}} & \cdots & \overrightarrow{\alpha_{n,l}} \end{bmatrix} + (1-t) \begin{bmatrix} \overrightarrow{\beta_{1,1}} & \overrightarrow{\beta_{1,2}} & \cdots & \overrightarrow{\beta_{1,l}} \\ \vdots & & & \vdots \\ \overrightarrow{\beta_{n,1}} & \overrightarrow{\beta_{n,2}} & \cdots & \overrightarrow{\beta_{n,l}} \end{bmatrix} \right) \begin{bmatrix} \overrightarrow{A_1^T} \\ \overrightarrow{A_2^T} \\ \vdots \\ \overrightarrow{A_l^T} \end{bmatrix},$$

with the degrees of the generic forms of $\overrightarrow{\alpha_{i,j}}$ and $\overrightarrow{\beta_{i,j}}$ chosen so that $\mathbf{h}_{\text{upp}}(t, d_1, \dots, d_n)$ is a system of n forms of degrees d_1, \dots, d_n whenever t is a random complex number. We denote the start points of $\mathbf{h}_{\text{upp}}(t; d_1, \dots, d_n)$ as $S_{\mathbf{h}_{\text{upp}}}$ and take them to be the isolated solutions of $\mathbf{h}_{\text{upp}}(1; d_1, \dots, d_n)$. Denote the end points of $\mathbf{h}_{\text{upp}}(t; d_1, \dots, d_n)$ as $T_{\mathbf{h}_{\text{upp}}}$.

With this definition, we have when $t = 1$ that $\mathbf{h}_{\text{upp}}(t; d_1, \dots, d_n)$ is a system of n \mathcal{I}' -generic forms f'_1, \dots, f'_n of degrees d_1, \dots, d_n . On the other hand, when $t = 0$ we have $\mathbf{h}_{\text{upp}}(t; d_1, \dots, d_n)$ is a system of n \mathcal{I} -generic forms of degrees d_1, \dots, d_n . By the fundamental theorem of parameter continuation of isolated roots [10] [Theorem 7.1.6] it follows that $T_{\mathbf{h}_{\text{upp}}}$ contains all isolated solutions of f_1, \dots, f_n . In particular, this proves Theorem 13 because $\#S_{\mathbf{h}_{\text{upp}}} \geq \#T_{\mathbf{h}_{\text{upp}}}$.

Theorem 13. *Let $\mathcal{I} \subset \mathbb{C}[x_0, \dots, x_n]$ be generated by the forms B_1, \dots, B_l such that $B_j = \sum_k A_{j,k}$ with $A_{j,k}$ being a term. If we let \mathcal{I}' be generated by $A_{j,k}$, then*

$$E_{\bullet}(\mathcal{I}'; d_1, \dots, d_n) \geq E_{\bullet}(\mathcal{I}; d_1, \dots, d_n).$$

Moreover, the parameter homotopy $\mathbf{h}_{\text{upp}}(t; d_1, \dots, d_n)$ has endpoints $T_{\mathbf{h}_{\text{upp}}}$ containing all isolated solutions of f'_1, \dots, f'_n .

Now that we have the theorem, we present our algorithm.

Input: Natural numbers d_1, \dots, d_n and generators B_1, \dots, B_l of an ideal \mathcal{I} in $\mathbb{C}[x_0, \dots, x_n]$ such that $B_j = \sum_k A_{j,k}$ and $A_{j,k}$ is a term.
Output : The excess number $E_{\bullet}(\mathcal{I}; d_1, \dots, d_n)$.
Step 1 : Construct the \mathbf{h}_{upp} -homotopy $\mathbf{h}_{\text{upp}}(t; d_1, \dots, d_n)$.
Step 2: Solve the start system $\mathbf{h}_{\text{upp}}(1; d_1, \dots, d_n) := [f'_1, \dots, f'_n]^T$ and compute $S_{\mathbf{h}_{\text{upp}}}$.
Step 3: Use the \mathbf{h}_{upp} -homotopy to determine $T_{\mathbf{h}_{\text{upp}}}$ and an upper bound of $E_{\bullet}(\mathcal{I}; d_1, \dots, d_n)$.
Step 4: Use a numerical membership test to determine $E_{\bullet}(\mathcal{I}; d_1, \dots, d_n)$ and the isolated solutions of $\mathbf{h}_{\text{upp}}(0; d_1, \dots, d_n) := f_1, \dots, f_n$.

For this algorithm, we assume in Step 2 that the excess intersection of a monomial ideal can be determined. While this may not always be true in practice, the first section has given some tools to address this case. We now give an example where \mathcal{I} defines the twisted cubic.

Example 14. Let the ideal $\mathcal{I} \subset \mathbb{C}[x, y, z, w]$ be generated by the forms

$$B_1 = z^2 - yw, B_2 = yz - xw, B_3 = y^2 - xz,$$

and suppose we want to calculate $E_{\bullet}(\mathcal{I}; 3, 3, 3)$. To run the first algorithm, we input $d_1 = d_2 =$

$d_3 = 3$ and B_1, B_2, B_3 . In Step 1, we determine $\mathbf{h}_{\text{upp}}(t; d_1, \dots, d_n) =$

$$\left(t \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{16} \\ a_{21} & a_{22} & \cdots & a_{26} \\ a_{31} & a_{32} & \cdots & a_{36} \end{bmatrix} + (1-t) \begin{bmatrix} b_{11} & b_{11} & b_{12} & b_{12} & b_{13} & b_{13} \\ b_{21} & b_{21} & b_{22} & b_{22} & b_{23} & b_{23} \\ b_{31} & b_{31} & b_{32} & b_{32} & b_{33} & b_{33} \end{bmatrix} \right) \begin{bmatrix} z^2 \\ -yw \\ yz \\ -xw \\ y^2 \\ -xz \end{bmatrix}.$$

The forms a_{ij} and b_{ij} are generic linear forms of $\mathbb{C}[x, y, z, w]$. Once we have solved the system $\mathbf{h}_{\text{upp}}(1; 3, 3, 3)$ in Step 2, we path track in Step 3 to calculate $T_{\mathbf{h}_{\text{upp}}}$ giving an upper bound $\#T_{\mathbf{h}_{\text{upp}}}$ of $E_{\bullet}(\mathcal{I}; 3, 3, 3)$. In Step 4, we use a numerical membership test [10] to conclude $E_{\bullet}(\mathcal{I}; 3, 3, 3) = 10$. Indeed, if

$$\begin{bmatrix} b_{11} \\ b_{12} \\ \vdots \\ b_{33} \end{bmatrix} = \begin{bmatrix} 1/2 & 1 & 4/5 & 1/3 & 1/5 & 7/8 & 13 & 1/3 \\ 3 & 7 & 9/7 & 1/8 & 4 & 1/6 & 5 & -1 \\ -5 & 4 & 7/8 & 8/9 & 3 & 1/15 & 1/6 & -8 \\ -1/4 & 2 & 1/3 & -1 & -1 & -2 & 7/9 & 1/4 \end{bmatrix}^T \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix},$$

then we find the ten excess points are $s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8 = \bar{s}_5, s_9 = \bar{s}_6, s_{10} = \bar{s}_7$:

	s_1	s_2	s_3	s_4
x	-6.1999	-0.2081	-1.0024	-0.1530
y	5.9766	0.5979	3.1208	0.3771
z	-2.3702	-2.1386	-5.1077	-0.6183
w	1	1	1	1

	s_5	s_6	s_7
x	$-.6493 + 1.4057i$	$0.4713 - 0.0461i$	$2.9076 + 0.0384i$
y	$.4134 - 1.4061i$	$0.2603 - 0.5271i$	$-1.0341 + 1.7553i$
z	$-1.1267 + 0.3173i$	$-0.9278 + 0.1923i$	$-0.7082 - 1.2392i$
w	1	1	1

3.2 Algorithm two and the \mathbf{h}_{ite} -homotopy

While our first algorithm was a probability-one algorithm to determine the excess number $E_{\bullet}(\mathcal{I}; d_1, \dots, d_n)$, the second algorithm is probabilistic. The algorithm uses the \mathbf{h}_{ite} -homotopy to compute a lower bound of excess numbers. By iterating this algorithm, the lower bounds can become sharp.

Definition 15. Let $B_1, \dots, B_l \in \mathbb{C}[x_0, \dots, x_n]$ be forms that generate \mathcal{I} and A_1, \dots, A_l be monomials that generate \mathcal{I}' such that $\deg B_j = \deg A_j$. The \mathbf{h}_{ite} -homotopy is defined as $\mathbf{h}_{\text{ite}}(t; d_1, \dots, d_n) :=$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1l} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nl} \end{bmatrix} \left(t \begin{bmatrix} A_1 \\ \vdots \\ A_l \end{bmatrix} + \gamma(1-t) \begin{bmatrix} B_1 \\ \vdots \\ B_l \end{bmatrix} \right),$$

with the degrees of the generic forms a_{ij} equal to $\deg f_i - \deg A_j$. We denote the start points of \mathbf{h}_{ite} as S_{ite} and take them to be the isolated solutions of $\mathbf{h}_{\text{ite}}(1; d_1, \dots, d_n)$ and denote the end points of $\mathbf{h}_{\text{ite}}(t; d_1, \dots, d_n)$ as T_{ite} .

With this definition, we have when $t = 1$ that $\mathbf{h}_{\text{ite}}(t; d_1, \dots, d_n)$ is a system of n \mathcal{I}' -generic forms f'_1, \dots, f'_n of degrees d_1, \dots, d_n . On the other hand, when $t = 0$ we have $\mathbf{h}_{\text{ite}}(t; d_1, \dots, d_n)$ is a system of n \mathcal{I} -generic forms of degrees d_1, \dots, d_n . While easy to set up, the \mathbf{h}_{ite} -homotopy is unable to use the fundamental theorem of parameter continuation of isolated roots [10] [Theorem 7.1.6] so $T_{\mathbf{h}_{\text{ite}}}$ does not necessarily contain all isolated solutions of f_1, \dots, f_n . However, after doing a membership test, we can determine some points in $T_{\mathbf{h}_{\text{ite}}}$ are isolated solutions of f_1, \dots, f_n . So what we have is a lower bound on $E_{\bullet}(\mathcal{I}; d_1, \dots, d_n)$. But, by iterating this homotopy, we can find more isolated solutions and give a better lower bound.

Input: Natural numbers d_1, \dots, d_n , generators B_1, \dots, B_l of an ideal $\mathcal{I} \subset \mathbb{C}[x_0, \dots, x_n]$, monomials A_1, A_2, \dots, A_l such that $\deg A_j = \deg B_j$, and a (possibly empty) set W of isolated solutions of f_1, \dots, f_n .

Output : A set W_{ite} containing W of isolated solutions of f_1, \dots, f_n , and $\#W_{\text{ite}}$ a lower bound for the excess number $E_{\bullet}(\mathcal{I}; d_1, \dots, d_n)$.

Step 1 : Construct the \mathbf{h}_{ite} -homotopy $\mathbf{h}_{\text{ite}}(t; d_1, \dots, d_n)$ and track start solutions S_{ite} to target solutions T_{ite} .

Step 2: Use a membership test to determine which solutions of T_{ite} are isolated and set W_{ite} to be the union of W and isolated solutions of T_{ite} .

Step 3: Output W_{ite} and $\#W_{\text{ite}}$ OR repeat steps 1 – 3 by making a different choice of γ in the \mathbf{h}_{ite} -homotopy.

To do numeric calculation, we often carry out computations in a straight line program. In this regard, the \mathbf{h}_{ite} -homotopy would be effective because we need only making changes in the constant γ to get new solutions. By taking different choices of γ in the \mathbf{h}_{ite} -homotopy we were able to produce the following example.

Example 16. If we take $A_1 = z^2, A_2 = yz, A_3 = y^2$, then we have the excess number $E_{\bullet}(\mathcal{I}; 3, 3, 3) = 7$. Next, we construct the \mathbf{h}_{ite} -homotopy as $\mathbf{h}_{\text{ite}}(t; d_1, \dots, d_n) =$

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{33} & b_{32} & b_{33} \end{bmatrix} \left(t \begin{bmatrix} z^2 \\ yz \\ y^2 \end{bmatrix} + \gamma(1-t) \begin{bmatrix} z^2 - yw \\ yz - xw \\ y^2 - xz \end{bmatrix} \right)$$

with a_{ij} the same as in Example 14. We find the 7 isolated solutions of $\mathbf{h}_{\text{ite}}(1; d_1, d_2, d_3)$ are $s'_1, s'_2, s'_3, s'_4, \bar{s}'_2, \bar{s}'_3, \bar{s}'_4$:

	s'_1	s'_2	s'_3	s'_4
x	-8.4814	-.0354 + .7868i	.8876 + .0702i	-.3053 + .4774i
y	8.2976	-.1201 + -.7446i	.3006 - .5880i	.3779 - .7007i
z	-2.9043	-.9638 + .1650i	-1.2929 + .2635i	-1.8276 + .6092i
w	1	1	1	1

By taking γ to be different random complex numbers and keeping b_{ij} fixed, with 4 iterations, we were able to find that $E_{\bullet}(\mathcal{I}; d_1, \dots, d_n)$ has a lower bound of 10. By the previous subsection, we know that this lower bound is actually sharp.

We comment that with certain choices of A_1, A_2, A_3 defining \mathcal{I} , it can happen that $E_\bullet(\mathcal{I}'; 3, 3, 3)$ is greater than, equal to, or less than $E_\bullet(\mathcal{I}; 3, 3, 3)$. So one may be tracking too many paths, too few paths, or perhaps luckily the right number. Open questions remain about for which choice of monomials A_1, \dots, A_l yield the best computational results. In addition, how should we choose γ to guarantee new solutions will be found as we iterate; and how can we verify that our lower bound has become sharp are also interesting questions. These questions will remain for future work, and their answers may depend heavily on the context of the problem.

Remark 17. We remark that the \mathbf{h}_{ite} -homotopy need not have had the A_j be monomials. Any choice of a form A_j whose degree equals B_j could have been used. However, in this section, we have made the assumption that excess intersections of monomial ideals can be computed effectively, as we saw combinatorics can be used to understand excess numbers of monomial ideals.

To conclude, we have shown that determining excess numbers of monomial ideals can be reduced to computing a mixed volume in certain cases. With this idea, we are able to provide an explicit formula for excess numbers of ideals with generic generators. We presented two algorithms using numerical algebraic geometry to determine excess numbers of any ideal. We also demonstrated that these algorithms have successfully lead to the calculation of excess numbers of an ideal defining the twisted cubic. We believe that ideals from combinatorics defined by sparse forms over many unknowns can have their excess numbers computed effectively using the \mathbf{h}_{upp} -homotopy, while the \mathbf{h}_{ite} -homotopy may be used in any situation.

References

- [1] Franz J. Király, Paul von Büna, Frank C. Meinecke, Duncan A.J. Blythe, and Klaus-Robert Müller. Algebraic geometric comparison of probability distributions. *J. Mach. Learn. Res.*, 13:855–903, March 2012.
- [2] Franz J. Király, Paul von Büna, Jan Saputra Müller, Duncan A. J. Blythe, Frank C. Meinecke, and Klaus-Robert Müller. Regression for sets of polynomial equations. *Journal of Machine Learning Research - Proceedings Track*, 22:628–637, 2012.
- [3] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.
- [4] Andrew J Sommes Daniel J. Bates, Jonathan D. Hauenstein and Charles W. Wampler. Bertini: Software for numerical algebraic geometry. Available at <http://www.nd.edu/~sommese/bertin>.
- [5] Daniel R. Grayson and Michael E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at <http://www.math.uiuc.edu/Macaulay2/>.
- [6] Daniel J. Bates, David Eklund, and Chris Peterson. Computing intersection numbers of Chern classes. *J. Symbolic Comput.*, 50:493–507, 2013.

- [7] Sandra Di Rocco, David Eklund, Chris Peterson, and Andrew J. Sommese. Chern numbers of smooth varieties via homotopy continuation and intersection theory. *J. Symb. Comput.*, 46(1):23–33, January 2011.
 - [8] Torgunn Karoline Moe and Nikolay Qviller. Segre classes on smooth projective toric varieties. arXiv:1204.4884, 2012.
 - [9] B. Huber and B. Sturmfels. Bernstein’s theorem in affine space. *Discrete Comput. Geom.*, 17(2):137–141, 1997.
 - [10] Andrew J. Sommese and Charles W. Wampler, II. *The numerical solution of systems of polynomials*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005. Arising in engineering and science.
 - [11] Günter M. Ziegler. *Lectures on polytopes*. Springer-Verlag, New York, 1995.
 - [12] Velleda Baldoni, Nicole Berline, Jesus A. De Loera, Matthias Köppe, and Michele Vergne. How to integrate a polynomial over a simplex. *Math. Comp.*, 80(273):297–325, 2011.
- (Jose Rodriguez) Department of Mathematics, University of California at Berkeley,
 970 Evans Hall 3840, Berkeley, CA 94720-3840 USA
E-mail address: jo.ro@berkeley.edu